

# SYMMETRIES OF DEGENERATE CENTER SINGULARITIES OF PLANE VECTOR FIELDS

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ABSTRACT. Let  $D^2 \subset \mathbb{R}^2$  be a closed unit 2-disk centered at the origin  $O \in \mathbb{R}^2$ , and  $F$  be a smooth vector field such that  $O$  is a unique singular point of  $F$  and all other orbits of  $F$  are simple closed curves wrapping once around  $O$ . Thus topologically  $O$  is a “center” singularity. Let  $\mathcal{D}^+(F)$  be the group of all diffeomorphisms of  $D^2$  which preserve orientation and orbits of  $F$ .

In the previous paper the author described the homotopy type of  $\mathcal{D}^+(F)$  under assumption that the 1-jet  $j^1 F(O)$  of  $F$  at  $O$  is non-degenerate. In this paper degenerate case of  $j^1 F(O)$  is considered. Under additional “non-degeneracy assumptions” on  $F$  the path components of  $\mathcal{D}^+(F)$  with respect to distinct weak topologies are described.

## 1. INTRODUCTION

Let  $D^2 = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$  be a closed unit 2-disk centered at the origin  $O \in \mathbb{R}^2$ ,  $V \subset \mathbb{R}^2$  be a closed subset diffeomorphic with  $D^2$ ,  $z \in \text{Int}V$ , and

$$F = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y}$$

be a  $C^\infty$  vector field on  $V$ . We will say that  $F$  is a *TC vector field on  $V$  with topological center at  $z$*  if it satisfies the following conditions:

- (T1)  $z$  is a unique singular point of  $F$ ,
- (T2)  $F$  is tangent to  $\partial V$ , so  $\partial V$  is an orbit of  $F$ , and
- (T3) all other orbits of  $F$  are closed.

If  $V = D^2$ , then we will always assume that  $F(O) = 0$ .

Let  $F$  be a TC vector field on  $D^2$ . Then it easily follows from Poincaré-Bendixson theorem, [PM], that there exists a *homeomorphism*

$$h = (h_1, h_2) : D^2 \rightarrow D^2$$

such that  $h(O) = O$  and for every other orbit  $o$  of  $F$  its image  $h(o)$  is the circle of some radius  $c \in (0, 1]$  centered at origin, see Figure 1. This motivates the term TC which we use, see [M10]. Moreover, it can be assumed that the restriction  $h : D^2 \setminus O \rightarrow D^2 \setminus O$  is a  $C^\infty$  diffeomorphism.

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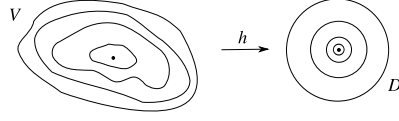


FIGURE 1.

**Definition 1.1.** A continuous function  $f : D^2 \rightarrow [0, 1]$  will be called a *first strong integral* for  $F$  if

- (i)  $f$  is  $C^\infty$  on  $D^2 \setminus O$  and has no critical points in  $D^2 \setminus O$ ,
- (ii)  $f^{-1}(0) = O$ ,  $f^{-1}(1) = \partial D^2$ , and for  $c \in [0, 1]$  the set  $f^{-1}(c)$  is an orbit of  $F$ .

Notice that we do not require that  $f$  is  $C^\infty$  at  $O$ . It also follows from (ii) that  $f$  takes distinct values on distinct orbits. For instance if  $h$  is  $C^\infty$  on  $D^2 \setminus O$ , then the function  $f = h_1^2 + h_2^2$  is the *first strong integral* for  $F$ .

Consider the following matrix

$$\nabla F = \begin{pmatrix} \frac{\partial F_1}{\partial x}(O) & \frac{\partial F_1}{\partial y}(O) \\ \frac{\partial F_2}{\partial x}(O) & \frac{\partial F_2}{\partial y}(O) \end{pmatrix}.$$

We will call  $\nabla F$  a *linear part* or *linearization* of  $F$  at  $O$ .

Suppose  $F$  is a TC vector field and  $\nabla F$  is non-degenerate. Then it can be shown that there are local coordinates at  $O$  in which

$$j^1 F(O) = -ay \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y}$$

for some  $a \neq 0$ , so the 1-jet of  $F$  at  $O$  is a “rotation”. This class of singularities is well-studied from many points of view, see e.g. [Po, Lp, Ta, B, Si1, Si2]. In particular, in [Ta] normal forms of such vector fields are obtained.

Denote by  $\mathcal{D}(F)$  the group of  $C^\infty$  diffeomorphisms  $h$  of  $D^2$  such that  $h(o) = o$  for each orbit  $o$  of  $F$ . Let also  $\mathcal{D}^+(F)$  be the subgroup of  $\mathcal{D}(F)$  consisting of all orientation preserving diffeomorphisms, and  $\mathcal{D}^\partial(F)$  be the subgroup of  $\mathcal{D}^+(F)$  consisting of all diffeomorphisms fixed of  $\partial D^2$ . We endow  $\mathcal{D}(F)$ ,  $\mathcal{D}^+(F)$ , and  $\mathcal{D}^\partial(F)$  with the weak  $W^\infty$  topologies, see §6.

The main result of [M10] describes the homotopy types of  $\mathcal{D}^\partial(F)$  and  $\mathcal{D}^+(F)$  for a TC vector field with non-degenerate  $\nabla F$ , see (i) of Proposition 2.6. Moreover, in [M7]

In this paper we study TC vector fields with degenerate  $\nabla F$  and describe path components of  $\mathcal{D}^+(F)$  and  $\mathcal{D}^\partial(F)$  under additional “non-degeneracy” assumptions, see Theorems 2.1, 2.3, and 2.8. The obtained results are not so complete as in the non-degenerate case due to the variety of normal forms.

Together with results of [M10] these theorems will be used in another paper to extend [M2] to a large class of  $C^\infty$  functions on surfaces.

## 2. FORMULATION OF RESULTS

Let  $F$  be a TC vector field on  $D^2$ . Denote by  $\mathcal{E}(F)$  the subset of  $C^\infty(D^2, D^2)$  which consists of all maps  $h : D^2 \rightarrow D^2$  satisfying the following conditions:

- $h(o) = o$  for every orbit  $o$  of  $F$ . In particular,  $h(O) = O$ .
- $h$  is a local diffeomorphism at  $O$ .

Evidently,  $\mathcal{E}(F)$  is a subsemigroup of  $C^\infty(D^2, D^2)$  with respect to the usual composition of maps. Consider the map

$$j : \mathcal{E}(F) \rightarrow \mathrm{GL}(2, \mathbb{R}), \quad j(h) = J(h, O),$$

associating to every  $h \in \mathcal{E}(F)$  its Jacobi matrix  $J(h, O)$  at  $O$ . Let

$$L(F) := j(\mathcal{E}(F))$$

be the image of  $j$ . Then a priori  $L(F)$  is a subsemigroup of  $\mathrm{GL}(2, \mathbb{R})$ .

Let  $\mathcal{E}^+(F) = j^{-1}(\mathrm{GL}^+(2, \mathbb{R}))$  be the subset of  $\mathcal{E}(F)$  consisting of all maps  $h$  with positive Jacobian at  $O$ .

Let also  $\mathcal{E}^\partial(F) \subset \mathcal{E}^+(F)$  be the subsemigroup consisting of all maps  $h$  fixed on  $\partial D^2$ , i.e.  $h(x) = x$  for all  $x \in \partial D^2$ . Evidently,

$$(2.1) \quad \mathcal{D}(F) \subset \mathcal{E}(F), \quad \mathcal{D}^+(F) \subset \mathcal{E}^+(F), \quad \mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F).$$

For  $r = 0, 1, \dots, \infty$  denote by  $\mathcal{E}(F)^r$ ,  $(\mathcal{E}^+(F))^r$ , etc.) the space  $\mathcal{E}(F)$ ,  $(\mathcal{E}^+(F))$ , etc.) endowed with weak  $W^r$  topology §6. Let also  $\mathcal{E}_{\mathrm{id}}(F)^r$ ,  $(\mathcal{E}_{\mathrm{id}}^+(F))^r$ , etc.) be the path component of the identity map  $\mathrm{id}_{D^2}$  in  $\mathcal{E}(F)$ ,  $(\mathcal{E}^+(F))$ , etc.) with respect to  $W^r$  topology at least for  $r \geq 1$ . Evidently, each  $h \in \mathcal{E}(F) \setminus \mathcal{E}^+(F)$  (if it exists) changes orientation of  $D^2$ , whence  $\mathcal{E}^+(F)^r$  consists of full path components of  $\mathcal{E}(F)^r$  for any  $r$ . In particular,

$$\mathcal{E}_{\mathrm{id}}^+(F)^r = \mathcal{E}_{\mathrm{id}}(F)^r \quad \forall r = 0, 1, \dots, \infty.$$

It turns out that it is more convenient to work with  $\mathcal{E}(F)$  instead of  $\mathcal{D}(F)$ . Moreover, the following theorem shows that such a replacement does not loose the information about homotopy types.

We will assume throughout that the identity map  $\mathrm{id}_{D^2}$  is a base point and therefore it will be omitted from notation. For instance we denote the  $n$ -th homotopy group  $\pi_n(\mathcal{E}(F)^r, \mathrm{id}_{D^2})$  simply by  $\pi_n \mathcal{E}(F)^r$  and so on.

**Theorem 2.1.** *Let  $F$  be a TC vector field on  $D^2$ . Let  $\mathcal{D}$  denotes one of the groups  $\mathcal{D}(F)$ ,  $\mathcal{D}^+(F)$ , or  $\mathcal{D}^\partial(F)$ , and  $\mathcal{E}$  be the corresponding semigroup  $\mathcal{E}(F)$ ,  $\mathcal{E}^+(F)$ , or  $\mathcal{E}^\partial(F)$ . By  $\mathcal{D}^r$  (resp.  $\mathcal{E}^r$ ) we denote the topological space  $\mathcal{D}$  (resp.  $\mathcal{E}$ ) endowed with  $W^r$  topology. Then*

- (1) *the inclusion  $\mathcal{D}^r \subset \mathcal{E}^r$  is a **weak homotopy equivalence**<sup>1</sup> for  $r \geq 1$ ;*

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<sup>1</sup>Recall that the map  $i : \mathcal{D} \rightarrow \mathcal{E}$  is a *weak homotopy equivalence* if for each  $n \geq 0$  the induced map  $i_n : \pi_n(\mathcal{D}, x) \rightarrow \pi_n(\mathcal{E}, x)$  of homotopy sets (groups for  $n \geq 1$ ) is a bijection for each  $x \in \mathcal{D}$ .

- (2) in  $W^0$  topology the induced map  $\pi_0 \mathcal{D}^0 \rightarrow \pi_0 \mathcal{E}^0$  is a **surjection**;
- (3) for each  $r \geq 0$  the semigroup  $\pi_0 \mathcal{E}^r$  is a **group** and any two path components of  $\mathcal{E}^r$  are homeomorphic each other.

**Remark 2.2.** In general a topological semigroup may have path components which are non homeomorphic each other. For instance this often so for the semigroup of continuous maps  $C(X, X)$  of a topological space  $X$  with non-trivial homotopy groups, see e.g. [Ha74].

The next result describes the relative homotopy groups of the pair  $(\mathcal{E}^+(F), \mathcal{E}^\partial(F))$ .

**Theorem 2.3.** *Let  $F$  be a TC vector field on  $D^2$ . Then for each  $r \geq 0$*

$$\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) = \begin{cases} \mathbb{Z}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the inclusion  $\mathcal{E}^\partial(F) \subset \mathcal{E}^+(F)$  yields **isomorphisms**

$$(2.2) \quad \pi_n \mathcal{E}^\partial(F)^r \rightarrow \pi_n \mathcal{E}^+(F)^r, \quad n \geq 2,$$

and we also have the following exact sequence:

$$(2.3) \quad 0 \rightarrow \pi_1 \mathcal{E}^\partial(F)^r \rightarrow \pi_1 \mathcal{E}^+(F)^r \rightarrow \mathbb{Z} \rightarrow \\ \rightarrow \pi_0 \mathcal{E}^\partial(F)^r \rightarrow \pi_0 \mathcal{E}^+(F)^r \rightarrow 0.$$

Our next aim (see Theorem 2.8 below) is to obtain some information about the homotopy groups of  $\mathcal{E}(F)$ ,  $\mathcal{E}^+(F)$  and  $\mathcal{E}^\partial(F)$ . First we recall some definitions and preliminary results.

**2.4. Shift map.** Let  $\mathbf{F} : D^2 \times \mathbb{R} \rightarrow D^2$  be the flow generated by  $F$  and

$$\varphi : C^\infty(D^2, \mathbb{R}) \rightarrow C^\infty(D^2, D^2)$$

be the map defined by

$$\varphi(\alpha)(z) = \mathbf{F}(z, \alpha(z))$$

for  $\alpha \in C^\infty(D^2, \mathbb{R})$  and  $z \in D^2$ . We will call  $\varphi$  the *shift map* along orbits of  $F$  and denote its image in  $C^\infty(D^2, D^2)$  by  $Sh(F)$ :

$$Sh(F) := \varphi(C^\infty(D^2, \mathbb{R})) \subset C^\infty(D^2, D^2).$$

**Lemma 2.5.** *The following inclusions hold true:*

$$(2.4) \quad Sh(F) \subset \mathcal{E}_{\text{id}}(F)^\infty \subset \dots \subset \mathcal{E}_{\text{id}}(F)^1 \subset \mathcal{E}_{\text{id}}(F)^0.$$

*If  $Sh(F) = \mathcal{E}_{\text{id}}(F)^r$  for some  $r = 0, 1, \dots, \infty$ , then*

$$(2.5) \quad \mathcal{D}_{\text{id}}(F)^\infty = \dots = \mathcal{D}_{\text{id}}(F)^r$$

*whence the identity maps  $\text{id} : \mathcal{D}(F)^\infty \rightarrow \mathcal{D}(F)^s$  and  $\text{id} : \mathcal{E}(F)^\infty \rightarrow \mathcal{E}(F)^s$  for  $s \geq r$  yield bijections:*

$$\pi_0 \mathcal{D}(F)^\infty \approx \dots \approx \pi_0 \mathcal{D}(F)^r, \quad \pi_0 \mathcal{E}(F)^\infty \approx \dots \approx \pi_0 \mathcal{E}(F)^r,$$

*Proof.* The first inclusion in (2.4) follows from [M1, Cor. 21] and the others are evident. The fact that (2.5) is implied by the assumption  $Sh(F) = \mathcal{E}_{\text{id}}(F)^r$  is proved in [M7].  $\square$

The following Proposition 2.6 and Example 2.7 describe some results about  $\ker j$ ,  $Sh(F)$ , and  $\mathcal{E}_{\text{id}}(F)^r$  for TC vector fields. Then most complete information are given in the case when  $\nabla F$  is non-degenerate and when  $F$  is a “reduced” Hamiltonian vector field of some homogeneous polynomial on  $\mathbb{R}^2$ .

**Proposition 2.6.** *Let  $F$  be a TC vector field on  $D^2$ .*

- (1) *If  $\nabla F = 0$ , then  $Sh(F) \subset \ker j$ .*
- (2) *Suppose that  $\nabla F$  is degenerate, but is not zero. Then there are local coordinates at  $O$  in which  $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$  for some  $a \in \mathbb{R} \setminus \{0\}$ . Define the following subsets of  $GL^+(2, \mathbb{R})$ :*

$$(2.6) \quad \begin{aligned} \mathcal{A}_{++} &= \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, d \in \mathbb{R} \right\}, & \mathcal{A}_{--} &= \left\{ \begin{pmatrix} -1 & d \\ 0 & -1 \end{pmatrix}, d \in \mathbb{R} \right\}, \\ \mathcal{A}_{+-} &= \left\{ \begin{pmatrix} 1 & d \\ 0 & -1 \end{pmatrix}, d \in \mathbb{R} \right\}, & \mathcal{A}_{-+} &= \left\{ \begin{pmatrix} -1 & d \\ 0 & 1 \end{pmatrix}, d \in \mathbb{R} \right\}, \\ \mathcal{A} &= \mathcal{A}_{++} \cup \mathcal{A}_{--} \cup \mathcal{A}_{+-} \cup \mathcal{A}_{-+}. \end{aligned}$$

*Then*

$$(2.7) \quad j(Sh(F)) = \mathcal{A}_{++}, \quad j(\mathcal{E}^+(F)) \subset \mathcal{A}_{++} \cup \mathcal{A}_{--}, \quad j(\mathcal{E}(F)) \subset \mathcal{A}.$$

- (3) *If  $\nabla F$  is non-degenerate, then there are local coordinates at  $O$  in which  $F$  is given by*

$$(2.8) \quad F(x, y) = \alpha(x, y) \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + \bar{X} \frac{\partial}{\partial x} + \bar{Y} \frac{\partial}{\partial y},$$

where  $\alpha$  is a  $C^\infty$  function such that  $\alpha(O) \neq 0$ , and  $\bar{X}, \bar{Y}$  are **flat** at  $O$ . Moreover,

$$Sh(F) = \mathcal{E}_{\text{id}}(F)^\infty = \dots = \mathcal{E}_{\text{id}}(F)^0 = \mathcal{E}^+(F) = j^{-1}(\text{SO}(2)),$$

$$\mathcal{D}_{\text{id}}(F)^\infty = \dots = \mathcal{D}_{\text{id}}(F)^0,$$

the inclusions  $\mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F)$  and  $\mathcal{D}^+(F) \subset \mathcal{E}^+(F)$  are homotopy equivalences with respect to  $\mathbf{W}^\infty$  topologies,  $\mathcal{D}^\partial(F)$  is contractible, and  $\mathcal{D}^+(F)$  is homotopy equivalent to the circle.

- (4) *Let  $\theta : D^2 \setminus O \rightarrow (0, +\infty)$  be the function associating to each  $z \in D^2 \setminus O$  its period  $\theta(z)$  with respect to  $F$ . Then  $\theta$  is  $C^\infty$  on  $D^2 \setminus O$  and we will call it the **period function**.*

*In the cases (1) and (2), i.e. when  $\nabla F$  is degenerate,  $\lim_{z \rightarrow O} \theta(z) = +\infty$  and thus  $\theta$  can not be even continuously extended to all of  $D^2$ . On the other hand in the case (3)  $\theta$  extends to a  $C^\infty$  function on all of  $D^2$ .*

*Proof.* Statement (1) is a particular case of [M6, Lm. 5.3]. (2) and (4) are established in [M10].

(3) Representation (2.8) is due to F. Takens [Ta], and all other statements are proved in [M10]. Actually, F. Takens shown that except

for (2.8) there is also an infinite series of normal forms for vector fields with “rotation as 1-jet”, but the orbits of these vector fields are non-closed, and so they are not TC.  $\square$

**Example 2.7.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a real homogeneous polynomial in two variables such that  $O \in \mathbb{R}^2$  is a unique critical point of  $f$  being its global minimum. Then we can write

$$(2.9) \quad f(x, y) = \prod_{j=1}^k Q_j^{\beta_j}(x, y),$$

where every  $Q_j$  is a positive definite quadratic form,  $\beta_j \geq 1$ , and

$$\frac{Q_j}{Q_{j'}} \neq \text{const for } j \neq j'.$$

Then it is easy to see that  $D = \prod_{j=1}^k Q_j^{\beta_j-1}$  is the greatest common divisor of partial derivatives  $f'_x$  and  $f'_y$ . Let  $G = -f'_y \frac{\partial}{\partial x} + f'_x \frac{\partial}{\partial y}$  be the Hamiltonian vector field of  $f$  and

$$F = -(f'_y/D) \frac{\partial}{\partial x} + (f'_x/D) \frac{\partial}{\partial y}.$$

Then the coordinate function of  $F$  are relatively prime in the ring  $\mathbb{R}[x, y]$ . We will call  $F$  the **reduced** Hamiltonian vector field for  $f$ .

Fix  $\varepsilon > 0$  and put  $V = f^{-1}[0, \varepsilon]$ . Then  $F$  is a TC vector field on  $V$  with singularity at  $O$ .

If  $k = 1$ , then  $\nabla F$  is non-degenerate and a description of  $\pi_0 \mathcal{E}^+(F)^\infty$  and  $\mathcal{E}_{\text{id}}(F)^\infty$  is given by (3) of Proposition 2.6.

If  $k \geq 2$ , then  $\nabla F = 0$ . In this case, see [M7, M9],

$$\ker j = Sh(F) = \mathcal{E}_{\text{id}}(F)^\infty = \dots = \mathcal{E}_{\text{id}}(F)^1 \neq \mathcal{E}_{\text{id}}(F)^0 = \mathcal{E}^+(F),$$

$\mathcal{E}_{\text{id}}(F)^\infty$  is contractible with respect to  $W^\infty$  topology, and  $\pi_0 \mathcal{E}^+(F)^\infty \approx \mathbb{Z}_{2n}$  for some  $n \geq 1$ .

Now we can formulate our last result Theorem 2.8. It gives some information about *weak* homotopy types of  $\mathcal{E}(F)$  and  $\mathcal{E}^\partial(F)$  under certain assumptions on  $F$ . The main assumption is the following one:

$$(2.10) \quad \ker j \subset Sh(F).$$

It means that for every  $h \in \mathcal{E}(F)$  with the identity map as 1-jet there exists a  $C^\infty$  shift function on all of  $D^2$ .

**Theorem 2.8.** Let  $F$  be a TC vector field on  $D^2$  such that  $\nabla F$  is degenerate and  $\ker j \subset Sh(F)$ . Let also  $r \geq 1$ . Then the following statements hold true.

(1) If  $\nabla F = 0$ , then  $Sh(F) = \ker j$ . Let  $\text{id} \in \text{GL}(2, \mathbb{R})$  be the unit matrix. If in addition the path component of  $\text{id}$  in the image  $L(F) = j(\mathcal{E}(F))$  of  $j$  coincides with  $\{\text{id}\}$  (e.g. when  $L(F)$  is discrete), then  $Sh(F) = \mathcal{E}_{\text{id}}(F)^1$ .

Hence  $j$  induces isomorphisms

$$(2.11) \quad \pi_0 \mathcal{E}^+(F)^r \approx L(F) \cap \mathrm{GL}^+(2, \mathbb{R}), \quad \pi_0 \mathcal{E}(F)^r \approx L(F).$$

(2) If  $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$  for some  $a \neq 0$ , then

$$(2.12) \quad Sh(F) = \mathcal{E}_{\mathrm{id}}(F)^\infty = \cdots = \mathcal{E}_{\mathrm{id}}(F)^1 = j^{-1}(\mathcal{A}_{++}),$$

whence  $j$  yields a monomorphism, see (2.6),

$$\pi_0 \mathcal{E}(F)^\infty \longrightarrow \pi_0 \mathcal{A} \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

(3) The inclusion  $\mathcal{E}_{\mathrm{id}}^\partial(F)^r \subset \mathcal{E}_{\mathrm{id}}^+(F)^r$  between the identity path components is a weak homotopy equivalence, whence from Theorem 2.3 we have the isomorphisms

$$\pi_n \mathcal{E}^\partial(F)^r \approx \pi_n \mathcal{E}^+(F)^r, \quad n \geq 1,$$

and following exact sequence:

$$(2.13) \quad 0 \rightarrow \mathbb{Z} \rightarrow \pi_0 \mathcal{E}^\partial(F)^r \rightarrow \pi_0 \mathcal{E}^+(F)^r \rightarrow 0.$$

(4) Suppose that the image  $L(F)$  of  $j$  is finite. Then  $\pi_0 \mathcal{E}^\partial(F)^r \approx \mathbb{Z}$ ,  $\pi_0 \mathcal{E}^+(F)^r \approx \mathbb{Z}_n$  for some  $n \geq 0$ , and (2.13) has the following form

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\mathrm{mod} \, n} \mathbb{Z}_n \rightarrow 0.$$

If  $\mathcal{E}(F) \neq \mathcal{E}^+(F)$ , then  $\pi_0 \mathcal{E}(F)^r \approx \mathbb{D}_n$  is a dihedral group.

The proof of Theorems 2.1, 2.3, and 2.8 will be given in §§9-11. All of them are based on results of [M9] described in §7 about existence and uniqueness of shift functions for deformations in  $\mathcal{E}^+(F)$ .

### 3. THE INCLUSION $\mathcal{D}(F) \subset \mathcal{E}(F)$

The aim of this section is to prove Lemma 3.1 which allows to change elements of  $\mathcal{E}(F)$  outside some neighbourhood of  $O$  to produce diffeomorphisms.

Let  $F$  be a TC vector field of  $D^2$  and  $f : D^2 \rightarrow [0, 1]$  be a first strong integral for  $F$ , see Definition 1.1. For every  $c \in (0, 1]$  put  $U_c = f^{-1}[0, c]$ . Then  $U_c$  is invariant with respect to  $F$ .

**Lemma 3.1.** *Let  $h \in \mathcal{E}(F)$ . Then there exists  $g \in \mathcal{D}(F)$  such that  $h = g$  on some neighbourhood of  $O$ .*

*Proof.* By definition  $h \in \mathcal{E}(F)$  is a diffeomorphism at  $O$ , whence there exists  $\varepsilon \in (0, 1/2)$  such that  $h : U_{2\varepsilon} \rightarrow U_{2\varepsilon}$  is a diffeomorphism. Fix any  $C^\infty$ -diffeomorphism  $\mu : [0, 2\varepsilon] \rightarrow [0, 1]$  such that  $\mu = \mathrm{id}$  on  $[0, \varepsilon]$ , see Figure 1.

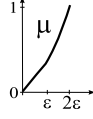


FIGURE 1.

We will now construct a diffeomorphism  $\psi : U_{2\varepsilon} \rightarrow D^2$  fixed on  $U_\varepsilon$  and such that  $f \circ \psi = \mu \circ f$ , i.e. it makes commutative the following diagram:

$$\begin{array}{ccc} U_{2\varepsilon} & \xrightarrow{\psi} & D^2 \\ f \downarrow & & \downarrow f \\ [0, 2\varepsilon] & \xrightarrow{\mu} & [0, 1] \end{array}$$

It follows that if  $c \in [0, 2\varepsilon]$  and  $o = f^{-1}(c)$  is an orbit of  $F$ , then  $\psi(o) = f^{-1}(\mu(c))$  is also an orbit of  $F$ . Then we can define a diffeomorphism  $g : D^2 \rightarrow D^2$  by

$$g = \psi \circ h|_{U_{2\varepsilon}} \circ \psi^{-1} : D^2 \xrightarrow{\psi^{-1}} U_{2\varepsilon} \xrightarrow{h} U_{2\varepsilon} \xrightarrow{\psi} D^2.$$

Then  $g \in \mathcal{D}(F)$  and since  $\psi$  is fixed on  $U_\varepsilon$ , it follows that  $g = h$  on  $U_\varepsilon$ .

Construction of  $\psi$  is similar to [M3, Lm. 5.1.3]. Consider the gradient vector field  $\nabla f$  of  $f$  defined on  $D^2 \setminus O$ , and let  $(\Phi_t)$  be the local flow of  $\nabla f$ . Let  $z \in U_{2\varepsilon}$  and  $\gamma$  be the orbit of  $z$  with respect to  $\Phi$ . Then  $\gamma$  intersect the level-set  $f^{-1}(\mu(f(z)))$  at a unique point  $\psi(z)$ , see Figure 2. Similarly to [M3] it can be shown that the correspondence  $z \mapsto \psi(z)$  is

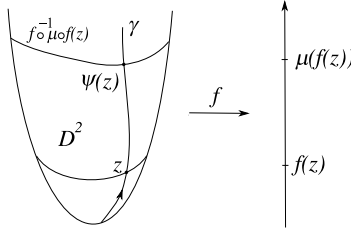


FIGURE 2.

a diffeomorphism of  $U_{2\varepsilon} \rightarrow D^2$  if and only if so is  $\mu$ .  $\square$

#### 4. SHIFT FUNCTIONS

Let  $M$  be a smooth ( $C^\infty$ ) manifold,  $F$  be a  $C^\infty$  vector field on  $M$  generating a flow  $\mathbf{F} : M \times \mathbb{R} \rightarrow M$ , and  $\varphi : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, M)$  be the *shift map along orbits of  $F$*  defined by  $\varphi(\alpha)(z) = \mathbf{F}(z, \alpha(z))$ .

If a subset  $V \subset M$ , a function  $\alpha : V \rightarrow \mathbb{R}$ , and a map  $h : M \rightarrow M$  are such that  $h(z) = \mathbf{F}(z, \alpha(z))$ , then we will say that  $\alpha$  is *shift function* for  $h$  on  $V$ , and that the restriction  $h|_V$  is in turn a *shift along orbits of  $F$  via  $\alpha$* .



For a  $C^\infty$  function  $\alpha : M \rightarrow \mathbb{R}$  we will denote by  $F(\alpha)$  the Lie derivative of  $\alpha$  along  $F$ .

**Lemma 4.1.** [M1, Th. 19] *Let  $V \subset M$  be an open subset,  $\alpha : V \rightarrow \mathbb{R}$  a  $C^\infty$  function, and  $h : V \rightarrow M$  be a map defined by  $h(z) = \mathbf{F}(z, \alpha(z))$ . Then  $h$  is a local diffeomorphism at some  $z_0 \in M$  if and only if  $F(\alpha)(z_0) \neq -1$ .*

**Lemma 4.2.** [M1] *Let  $\alpha_g, \alpha_h, \alpha_k : M \rightarrow \mathbb{R}$  be  $C^\infty$  functions and*

$$g = \varphi(\alpha_g), \quad h = \varphi(\alpha_h), \quad k = \varphi(\alpha_k)$$

*be the corresponding shifts. Suppose also that  $k$  is a diffeomorphism. Then the following functions*

$$\alpha_{g \circ h} = \alpha_g \circ h + \alpha_h, \quad \alpha_{k^{-1}} = -\alpha_k \circ k^{-1},$$

$$\alpha_{g \circ k^{-1}} = (\alpha_g - \alpha_k) \circ k^{-1}$$

*are  $C^\infty$  shift functions for  $g \circ h$ ,  $k^{-1}$ , and  $g \circ k^{-1}$  respectively.*

*Proof.* Formulae for  $\alpha_{g \circ h}$  and  $\alpha_{k^{-1}}$  coincide with [M1, Eq. (8),(9)]. They also imply formula for  $\alpha_{g \circ k^{-1}}$ .  $\square$

## 5. SHIFT FUNCTIONS FOR $\mathcal{E}^+(F)$

Let  $B = \{(\phi, r) \in \mathbb{R}^2 : 0 \leq r \leq 1\}$  be a closed strip,

$$\check{B} = \{(\phi, r) \in \mathbb{R}^2 : 0 < r \leq 1\} = B \setminus \{r = 0\}$$

be a half-closed strip in  $\mathbb{R}^2$ , and  $P : B \rightarrow D^2$  be the map given by

$$P(r, \phi) = (r \cos \phi, r \sin \phi).$$

Then  $P(\check{B}) = D^2 \setminus O$  and the restriction  $P : \check{B} \rightarrow D^2 \setminus O$  is  $\mathbb{Z}$ -covering map such that the corresponding group of covering transformations is generated by the following map

$$\eta : B \rightarrow B, \quad \eta(\phi, r) = (\phi + 2\pi, r).$$

It follows that every  $C^\infty$  map  $h : D^2 \setminus O \rightarrow D^2 \setminus O$  lifts to a  $P$ -equivariant (i.e. commuting with  $\eta$ ) map  $\tilde{h} : \check{B} \rightarrow \check{B}$  such that

$$P \circ \tilde{h} = h \circ P.$$

Such  $\tilde{h}$  is not unique and can be replaced with  $\tilde{h} \circ \eta^n = \eta^n \circ \tilde{h}$  for any  $n \in \mathbb{Z}$ .

**Remark 5.1.** *It is well-known that if  $h : D^2 \rightarrow D^2$  is a  $C^\infty$  map being a local diffeomorphism at  $O$ , and such that  $h^{-1}(O) = O$ , then  $\tilde{h}$  extends to a  $C^\infty$  map  $\tilde{h} : B \rightarrow B$  being a diffeomorphism near  $\phi$ -axis  $\{r = 0\}$ . We will not use this fact in the present paper.*

Let  $F$  be a TC vector field on  $D^2$ . Since  $F$  is non-singular on  $D^2 \setminus O$ ,  $F$  lifts to a unique vector field  $G$  on  $\check{B}$  such that  $F \circ P = TP \circ G$ .

It is easy to see that every orbit  $\tilde{o}$  of  $G$  is non-closed, its image  $o = P(\tilde{o})$  is an orbit of  $F$ , and the map  $P : \tilde{o} \rightarrow o$  is a  $\mathbb{Z}$ -covering map.

Let  $\mathbf{G} : \check{B} \times \mathbb{R} \rightarrow \check{B}$  be the flow generated by  $G$ , then we have the following commutative diagram:

$$(5.1) \quad \begin{array}{ccc} \check{B} \times \mathbb{R} & \xrightarrow{\mathbf{G}} & \check{B} \\ P \times \text{id}_{\mathbb{R}} \downarrow & & \downarrow P \\ (D^2 \setminus O) \times \mathbb{R} & \xrightarrow{\mathbf{F}} & D^2 \setminus O \end{array}$$

In other words,  $\mathbf{F}_t \circ P(\tilde{z}) = P \circ \mathbf{G}_t(\tilde{z})$  for all  $\tilde{z} \in \check{B}$  and  $t \in \mathbb{R}$ .

In particular, if  $\alpha : D^2 \rightarrow \mathbb{R}$  is a  $C^\infty$  function and  $h = \varphi(\alpha)$ , i.e.  $h(z) = \mathbf{F}(z, \alpha(z))$ , then the map  $\tilde{h} : \check{B} \rightarrow \check{B}$  given by  $\mathbf{G}(\tilde{z}, \alpha \circ P(\tilde{z}))$  is a lifting of  $h$ . Indeed,

$$(5.2) \quad h \circ P(\tilde{z}) = \mathbf{F}(P(\tilde{z}), \alpha \circ P(\tilde{z})) = P \circ \mathbf{G}(\tilde{z}, \alpha \circ P(\tilde{z})) = P \circ \tilde{h}(\tilde{z}).$$

**Lemma 5.2.** *Let  $h \in \mathcal{E}^+(F)$ . Then there exists a  $C^\infty$  shift function  $\beta : D^2 \setminus O \rightarrow \mathbb{R}$  for  $h$  on  $D^2 \setminus O$ , i.e.  $h(z) = \mathbf{F}(z, \beta(z))$  for  $z \in D^2 \setminus O$ . Any other  $C^\infty$  shift function for  $h$  on  $D^2 \setminus O$  is given by  $\beta + n\theta$  for some  $n \in \mathbb{Z}$ .*

*If  $\nabla F$  is degenerate, then any  $h \in \mathcal{E}^+(F)$  has at most one  $C^\infty$  shift function defined on all of  $D^2$ .*

*Proof.* By definition  $h^{-1}(O) = O$  and  $h$  is a local diffeomorphism at  $O$ . Then, as noted above, there exists a  $C^\infty$  lifting  $\tilde{h} : \check{B} \rightarrow \check{B}$  of  $h$  such that  $P \circ \tilde{h} = h \circ P$ . Moreover,  $h$  preserves orbits of  $F$ , whence  $h(\tilde{o}) = \tilde{o}$  for each orbit  $\tilde{o}$  of  $G$ .

Since the orbits of  $G$  are non-closed there exists a unique  $C^\infty$  shift function  $\tilde{\beta} : \check{B} \rightarrow \mathbb{R}$  for  $\tilde{h}$ , i.e.  $\tilde{h}(\tilde{z}) = \mathbf{G}(\tilde{z}, \tilde{\beta}(\tilde{z}))$  for all  $\tilde{z} \in \check{B}$ . Also notice that  $\mathbf{G}_t$  and  $\tilde{h}$  are  $\mathbb{Z}$ -equivariant. This easily implies that  $\tilde{\beta}$  is  $\mathbb{Z}$ -invariant, whence it defines a unique  $C^\infty$  function  $\beta : D^2 \setminus O \rightarrow \mathbb{R}$  such that  $\tilde{\beta} = \beta \circ P$ . Then it follows from (5.2) that  $\beta$  is a shift function for  $h$  with respect to  $F$ .

Suppose that  $\alpha : D^2 \setminus O \rightarrow \mathbb{R}$  is another  $C^\infty$  shift function for  $h$  on  $D^2 \setminus O$ . Then  $h(z) = \mathbf{F}(z, \alpha(z)) = \mathbf{F}(z, \beta(z))$  for  $z \neq 0$ , whence the difference  $\alpha(z) - \beta(z)$  is a certain integer multiple of the period  $\theta(z)$  of  $z$ . Since  $\theta$  and  $\alpha - \beta$  are  $C^\infty$  on  $D^2 \setminus O$ , it follows that  $\alpha - \beta = n\theta$  for some  $n \in \mathbb{Z}$ .

Finally, suppose that  $\nabla F$  is degenerate, and  $\alpha, \beta : D^2 \rightarrow \mathbb{R}$  are two  $C^\infty$  shift function for  $h$  defined on all of  $D^2$ . Then they also shift functions for  $h$  on  $D^2 \setminus O$ , whence  $\alpha - \beta = n\theta$  for some  $n \in \mathbb{Z}$ . But by Proposition 2.6  $\lim_{z \rightarrow O} \theta(z) = +\infty$ , while  $\alpha - \beta$  is  $C^\infty$  on all of  $D^2$ . Hence  $n = 0$ , i.e.  $\alpha = \beta$ .  $\square$

6.  $(\mathbf{K}, r)$ -DEFORMATIONS

Let  $A, B$  be smooth manifolds. Then the space  $C^\infty(A, B)$  admits a series  $\{\mathbf{W}^r\}_{r=0}^\infty$  of *weak* topologies, see [Hi]. Topology  $\mathbf{W}^0$  coincides with the *compact open* one. Let  $J^r(A, B)$ ,  $(r < \infty)$ , be the manifold of  $r$ -jets of maps  $A \rightarrow B$ . Then there is a natural inclusion  $i_r : C^\infty(A, B) \subset C^\infty(A, J^r(A, B))$  associating to each  $f : A \rightarrow B$  its  $r$ -jet prolongation  $j^r(f) : A \rightarrow J^r(A, B)$ . Endow  $C^\infty(A, J^r(A, B))$  with  $\mathbf{W}^0$  topology. Then the topology on  $C^\infty(A, B)$  induced by  $i_r$  is called  $\mathbf{W}^r$  topology. Finally, the topology  $\mathbf{W}^\infty$  is generated by all  $\mathbf{W}^r$  for  $0 \leq r < \infty$ .

Let  $\mathcal{X} \subset C^\infty(A, B)$  be a subset,  $\mathbf{K}$  be a Hausdorff, locally compact topological space, and  $\omega : \mathbf{K} \rightarrow \mathcal{X}$  be a map. Then  $\omega$  induces the following mapping  $\Omega : \mathbf{K} \times A \rightarrow B$  defined by  $\Omega(a, k) = \omega(k)(a)$ . Conversely, every map  $\Omega : \mathbf{K} \times A \rightarrow B$  such that  $\Omega(k, \cdot) : A \rightarrow B$  belongs to  $\mathcal{X}$  induces a map  $\omega : \mathbf{K} \rightarrow \mathcal{X}$ .

Endow  $\mathcal{X}$  with the induced  $\mathbf{W}^0$  topology. Then it is well known, e.g. [1, §44.IV], that  $\omega$  is continuous if and only if  $\Omega$  is so.

**Definition 6.1.** Let  $r = 0, \dots, \infty$ . Then a map  $\Omega : \mathbf{K} \times A \rightarrow B$  will be called a  $(\mathbf{K}, r)$ -*deformation in  $\mathcal{X}$*  if  $\Omega_k \in \mathcal{X}$  for all  $k \in \mathbf{K}$  and the induced map  $\omega : \mathbf{K} \rightarrow \mathcal{X}$  is continuous whenever  $\mathcal{X}$  is endowed with  $\mathbf{W}^r$  topology. In other words the map  $j^r : \mathbf{K} \times A \rightarrow J^r(A, B)$  associating to each  $(k, a) \in \mathbf{K} \times A$  the  $r$ -jet prolongation  $j^r \Omega_k(a)$  of  $\Omega_k$  at  $a$  is continuous.

If  $\mathbf{K} = [0, 1]$  then a  $(\mathbf{K}, r)$ -deformation will be called an  $r$ -*homotopy*.

7. SHIFT FUNCTIONS FOR  $(\mathbf{K}, r)$ -DEFORMATIONS

Let  $\mathbf{K}$  be a Hausdorff, locally compact, and path connected topological space,  $F$  be a TC vector field on  $D^2$ . Let also

$$\omega : \mathbf{K} \rightarrow \mathcal{E}^+(F)^r$$

be a continuous map into some  $\mathbf{W}^r$  topology of  $\mathcal{E}^+(F)$ , and

$$(7.1) \quad \Omega : \mathbf{K} \times D^2 \rightarrow D^2, \quad \Omega(k, z) = \omega(k)(z).$$

be the corresponding  $(\mathbf{K}, r)$ -deformation in  $\mathcal{E}^+(F)$ , so  $\Omega_k \in \mathcal{E}^+(F)$  for all  $k \in \mathbf{K}$ .

Then by Lemma 5.2 for each  $k \in \mathbf{K}$  the map  $\Omega_k$  has a (not unique)  $C^\infty$  shift function  $\Lambda_k$  defined on  $D^2 \setminus O$ . Thus we can define a map  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  by  $\Lambda(k, z) = \Lambda_k(z)$  which in general is not even continuous, though it is  $C^\infty$  for each  $k$ .

**Definition 7.1.** A  $(\mathbf{K}, r)$ -deformation  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  satisfying

$$(7.2) \quad \Omega(k, z) = \mathbf{F}(z, \Lambda(k, z)) \quad \forall (k, z) \in \mathbf{K} \times (D^2 \setminus O).$$

will be called a **shift function** for the  $(\mathbf{K}, r)$ -deformation  $\Omega$ .

The following lemma is a particular case of results of [M9], see also [M1, Th. 25]. It describes existence and uniqueness of shift functions for  $(\mathbf{K}, r)$ -deformations.

**Lemma 7.2.** [M9] *Let  $k_0 \in \mathbf{K}$  and  $\Lambda_{k_0}$  be any  $C^\infty$  shift function for  $\Omega_{k_0}$ . Then there exists at most one shift function  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  for  $\Omega$  such that  $\Lambda(k_0, z) = \Lambda_{k_0}(z)$ .*

*Moreover, if  $\mathbf{K}$  is simply connected, i.e.  $\pi_1 \mathbf{K} = 0$ , then any shift function  $\Lambda_{k_0}$  for  $\Omega_{k_0}$  uniquely extends to a shift function*

$$\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$$

*for  $\Omega$ .* □

This lemma will be used in the proof of Theorems 2.1 and 2.3. For the proof of Theorem 2.8 we will also need the following Lemmas 7.3 and 7.4.

Suppose now that  $\omega(\mathbf{K}) \subset Sh(F)$ , that is for each  $k \in \mathbf{K}$  the map  $\Omega_k$  has a  $C^\infty$  shift function defined on all of  $D^2$ . Let  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  be a shift function for  $\Omega$  such that  $\Lambda_{k_0}$  for some  $k_0 \in \mathbf{K}$  smoothly extends to all of  $D^2$ . The following lemma gives sufficient conditions when every other shift function  $\Lambda_k = \Lambda(k, \cdot)$  smoothly extends to all of  $D^2$ . Again it is a particular case of results of [M9].

**Lemma 7.3.** [M9] *Let  $\Omega : \mathbf{K} \times D^2 \rightarrow D^2$  be a  $(\mathbf{K}, r)$ -deformation admitting a shift function  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$ . Suppose that*

- (i)  $\ker j \subset Sh(F)$ ,
- (ii)  $\Omega_0 = \text{id}_{D^2}$  for some  $k_0 \in \mathbf{K}$ , and
- (iii)  $j(\Omega_k) = \text{id}$ , i.e.  $\Omega_k \in \ker j \subset Sh(F)$ , for all  $k \in \mathbf{K}$ .

*Then for each  $k \in \mathbf{K}$  the function  $\Lambda_k : D^2 \setminus O \rightarrow \mathbb{R}$  extends to a  $C^\infty$  function on all of  $D^2$ , though the induced function  $\Lambda : \mathbf{K} \times D^2 \rightarrow \mathbb{R}$  is not necessarily continuous.* □

Finally, we present a sufficient condition when a map into  $Sh(F)$  can be deformed into  $\ker j$ .

**Lemma 7.4.** *Let  $\mathbf{K}$  be path connected and simply connected,  $r \geq 1$ , and  $\omega : \mathbf{K} \rightarrow Sh(F)$  be a continuous map into  $W^r$  topology of  $Sh(F)$ . Suppose  $\nabla F$  is degenerate and  $\ker j \subset Sh(F)$ . Then there exists a homotopy  $\mathbf{B} : I \times \mathbf{K} \rightarrow Sh(F)$  such that  $\mathbf{B}_0 = \omega$ ,  $\mathbf{B}_1(\mathbf{K}) \subset \ker j$ , and  $\mathbf{B}_t(k) = \omega(k)$  for all  $k$  such that  $\omega(k) \in \ker j$ .*

*Proof.* If  $\nabla F = 0$ , then  $Sh(F) = \ker j$  and there is nothing to prove.

Suppose that  $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$  for some  $a \neq 0$ . Let  $\Omega : \mathbf{K} \times D^2 \rightarrow D^2$  be the corresponding  $(\mathbf{K}, r)$ -deformation in  $Sh(F)$ . Then

$$j(\Omega_k) = \begin{pmatrix} 1 & a\tau(k) \\ 0 & 1 \end{pmatrix}, \quad k \in I,$$

for some  $\tau(k) \in \mathbb{R}$ . Since  $\Omega$  is an  $r$ -homotopy with  $r \geq 1$ , it follows that the function  $\tau : \mathbf{K} \rightarrow \mathbb{R}$  is continuous. Moreover,  $\tau(k) = 0$  if and only if  $\Omega_k \in \ker j$ .

Define the homotopy  $\mathbf{B} : I \times \mathbf{K} \rightarrow Sh(F)$  by

$$\mathbf{B}(t, k)(z) = \mathbf{F}(\Omega(k, z), -t\tau(k)).$$

Then it is easy to see that  $\mathbf{B}$  satisfies the statement of our lemma.  $\square$

## 8. DEFORMATIONS IN $\mathcal{E}^+(F)$

In this section we prove the key Proposition 8.1 which will imply Theorems 2.1, 2.3, and 2.8.

Let  $\mathbf{K}$  be a Hausdorff, locally compact topological space and

$$\omega : \mathbf{K} \rightarrow \mathcal{E}^+(F)^r$$

be a continuous map into some  $W^r$  topology of  $\mathcal{E}^+(F)$ . Our aim is to show that under certain mild assumptions  $\omega$  is homotopic to a map into  $\mathcal{D}^\partial(F) = \mathcal{D}^+(F) \cap \mathcal{E}^\partial(F)$  so that the intersections of  $\omega(\mathbf{K})$  with  $\mathcal{D}^\partial(F)$ ,  $\mathcal{D}^+(F)$  and  $\mathcal{E}^\partial(F)$  remain in the corresponding spaces during all the homotopy. More precisely the following result holds true:

**Proposition 8.1.** *Suppose that either*

- (i)  $\mathbf{K}$  is a point and  $r \geq 0$ , or
- (ii)  $\mathbf{K}$  is compact, path connected, and simply connected, and  $r \geq 1$ .

*Let  $\mathbf{L} \subset \mathbf{K}$  be a (possibly empty) subset such that  $\omega(\mathbf{L}) \subset \mathcal{D}^+(F)$ , and  $\mathbf{P} \subset \mathbf{K}$  be a connected subset such that  $\omega(\mathbf{P}) \subset \mathcal{E}^\partial(F)$ . Thus we can regard  $\omega$  as a map of triples*

$$\omega : (\mathbf{K}; \mathbf{L}, \mathbf{P}') \rightarrow (\mathcal{E}^+(F)^r; \mathcal{D}^+(F)^r, \mathcal{E}^\partial(F)^r).$$

*Then there exists a homotopy of triples*

$$\mathbf{A}_t : (\mathbf{K}; \mathbf{L}, \mathbf{P}') \rightarrow (\mathcal{E}^+(F)^r; \mathcal{D}^+(F)^r, \mathcal{E}^\partial(F)^r), \quad t \in I,$$

*such that*

$$(8.1) \quad \mathbf{A}_0 = \omega \quad \text{and} \quad \mathbf{A}_1(\mathbf{K}) \subset \mathcal{D}^\partial(F).$$

*The phrase **homotopy of triples** means that*

$$(8.2) \quad \mathbf{A}_t(\mathbf{L}) \subset \mathcal{D}^+(F), \quad \mathbf{A}_t(\mathbf{P}) \subset \mathcal{E}^\partial(F),$$

*and therefore  $\mathbf{A}_t(\mathbf{L} \cap \mathbf{P}) \subset \mathcal{D}^+(F) \cap \mathcal{E}^\partial(F) = \mathcal{D}^\partial(F)$  for all  $t \in I$ .*

The proof will be given at the end of this section. Let

$$\Omega : \mathbf{K} \times D^2 \rightarrow D^2, \quad \Omega(k, z) = \omega(k)(z)$$

be the corresponding  $(\mathbf{K}, r)$ -deformation in  $\mathcal{E}^+(F)$ . Then by Lemma 7.2 there exists a shift function  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  for  $\Omega$ . The deformation of  $\Omega_k$  we will be produced via a deformation of  $\Lambda_k$ .

Let  $a, b \in (0, 1)$  be such that  $a < b$ ,  $f : D^2 \rightarrow [0, 1]$  be the first strong integral for  $F$ , see Definition 1.1, and  $\nu : [0, 1] \rightarrow [0, 1]$  be a  $C^\infty$  function such that  $\nu[0, a] = 1$  and  $\nu[b, 1] = 0$ . Define the following function  $\alpha : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  by

$$(8.3) \quad \alpha(k, z) = \nu \circ f(z) \cdot \Lambda(k, z), \quad (k, z) \in \mathbf{K} \times (D^2 \setminus O),$$

the map  $\Omega' : \mathbf{K} \times D^2 \rightarrow D^2$  by

$$(8.4) \quad \Omega'(k, z) = \begin{cases} \mathbf{F}(z, \alpha(k, z)), & z \neq O, \\ O, & z = O, \end{cases}$$

and a homotopy  $A : I \times \mathbf{K} \times D^2 \rightarrow D^2$  by

$$(8.5) \quad A(t, k, z) = \begin{cases} \mathbf{F}(z, (1-t)\alpha(k, z) + t\Lambda(k, z)), & z \neq O, \\ O, & z = O. \end{cases}$$

**Lemma 8.2.** *For  $t, k \in I \times \mathbf{K}$  denote  $A_t = A(t, \cdot, \cdot) : \mathbf{K} \times D^2 \rightarrow D^2$  and  $A_{t,k} = A(t, k, \cdot) : D^2 \rightarrow D^2$ . Then*

- (a)  $A_0 = \Omega$ ,  $A_1 = \Omega'$ , and  $A_t$  is a  $(\mathbf{K}, r)$ -deformation in  $\mathcal{E}(F)$  for each  $t \in I$ .
- (b)  $\Omega'_k$  is fixed on  $D^2 \setminus U_b$  for all  $k \in \mathbf{K}$ . In particular,  $A_1 = \Omega'$  is a deformation in  $\mathcal{E}^\partial(F)$ .
- (c) If for some  $(k, z) \in \mathbf{K} \times D^2$  the map  $\Omega_k$  is a local diffeomorphism at  $z$ , then so is  $A_{t,k}$  for each  $t \in I$ .
- (d) Denote  $\mathbf{Z} = \Lambda^{-1}(0) \subset \mathbf{K} \times (D^2 \setminus O)$ . Thus  $\Omega(k, z) = z$  for all  $(k, z) \in \mathbf{Z}$ . Then

$$A(t, k, z) = z, \quad \forall t \in I, (k, z) \in \mathbf{Z}.$$

- (e) Let  $\mathbf{P} \subset \mathbf{K}$  be a connected subset such that  $\Omega_k$  is fixed on  $\partial D^2$  for each  $k \in \mathbf{P}$  and  $\Lambda_{k_0}|_{\partial D^2} = 0$  for some  $k_0 \in \mathbf{P}$ . Then  $A_{t,k}$  is also fixed on  $\partial D^2$  for all  $(t, k) \in I \times \mathbf{P}$ .

Thus  $A$  induces a homotopy

$$(8.6) \quad \mathbf{A}_t : \mathbf{K} \rightarrow \mathcal{E}^+(F)^r, \quad \mathbf{A}_t(k)(z) = A(t, k, z)$$

such that  $\mathbf{A}_0 = \omega$ , and  $\mathbf{A}_1(\mathbf{K}) \subset \mathcal{E}^\partial(F)$ .

*Proof.* Statements (a) and (b) follows from (8.3)-(8.5).

(c) Denote

$$(8.7) \quad \beta_{t,k}(z) = (1-t)\alpha(k, z) + t\Lambda(k, z) = ((1-t)\nu \circ f(z) + t) \cdot \Lambda_k(z).$$

Then by (8.5)  $\beta_{t,k}$  is a shift function for  $A_{t,k}$  on  $D^2 \setminus O$ .

The assumption that  $\Omega_k$  is a local diffeomorphism at  $z$  means that

$$(8.8) \quad F(\Lambda_k)(z) > -1,$$

see Lemma 4.1. Therefore by that lemma it suffices to verify that  $F(\beta_{t,k})(z) > -1$  for all  $t \in I$ .

Notice that

$$F(((1-t)\nu \circ f + t) \cdot \Lambda_k) = (1-t)F(\nu \circ f) + ((1-t)\nu \circ f + t)F(\Lambda_k).$$

The first summand is zero since  $f$  and therefore  $\nu \circ f$  are constant along orbits of  $F$ . Moreover,  $0 \leq \nu(z) \leq 1$ , whence we get from (8.8) that the second summand is  $> -1$ . Hence  $F(\beta_{t,k})(z) > -1$  for all  $k \in \mathbf{K}$ .

- (d) If  $\Lambda(k, z) = 0$  for some  $(k, z) \in \mathbf{K} \times (D^2 \setminus O)$ , then by (8.7)  $\beta_{t,k}(z) = 0$ , whence  $A(t, k, z) = \mathbf{F}(z, \beta_{t,k}(z)) = \mathbf{F}(z, 0) = z$ .

(e) If  $\Omega_k$  is fixed on  $\partial D^2$  for some  $k \in \mathbf{K}$ , then  $\Lambda_k$  takes on  $\partial D^2$  a constant value:

$$\Lambda_k(\partial D^2) = n_k \cdot \theta(\partial D^2)$$

for some  $n_k \in \mathbb{Z}$ . Since  $\mathbf{P}$  is connected,  $\Lambda$  is continuous on  $\mathbf{P} \times \partial D^2$ , and possible values of  $\Lambda_k$  on  $\partial D^2$  are discrete, it follows that  $\Lambda$  is constant on  $\mathbf{P} \times \partial D^2$ . In particular,  $\Lambda|_{\mathbf{P} \times \partial D^2} = \Lambda_{k_0}|_{\mathbf{P} \times \partial D^2} = 0$ . Then by (d)  $A_{t,k}$  is fixed on  $\partial D^2$  for all  $(k, t) \in I \times \mathbf{P}$ .  $\square$

**Proof of Proposition 8.1.** We will find  $a, b \in (0, 1)$  and a shift function  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  for  $\Omega$  such that the corresponding homotopy  $\mathbf{A}_t$  constructed in Lemma 8.2 will satisfy (8.1) and (8.2).

*Choice of  $\Lambda$ .* Let  $\Lambda' : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  be any shift function for  $\Omega$ . Since  $\Omega_{k_0}$  is fixed on  $\partial D^2$  for some  $k_0 \in \mathbf{P}$ , we have that  $\Lambda'_{k_0}|_{\partial D^2} = n\theta(\partial D^2)$  for some  $n \in \mathbb{Z}$ .

Define another function  $\Lambda : \mathbf{K} \times (D^2 \setminus O) \rightarrow \mathbb{R}$  by

$$\Lambda(k, z) = \Lambda'(k, z) - n\theta(\partial D^2).$$

Then  $\Lambda$  is also a shift function for  $\Omega$  in the sense of (7.2) and satisfies

$$(8.9) \quad \Lambda_{k_0}|_{\partial D^2} = 0.$$

*Choice of  $a, b \in (0, 1)$ .* Notice that  $\Omega_k(U_b) = U_b$  for all  $k \in \mathbf{K}$  and  $b \in (0, 1]$ . We claim that *there exists  $b \in (0, 1)$  such that the map  $\Omega_k : U_b \rightarrow U_b$  is a diffeomorphism for all  $k \in I$* . Indeed, by definition of  $\mathcal{E}^+(F)$  the map  $\Omega_k$  is a diffeomorphism at  $O$  for each  $k \in \mathbf{K}$ . This implies existence of  $b$  in the case (i), i.e. when  $\mathbf{K}$  is a point. In the case (ii) the assumption  $r \geq 1$  means that the partial derivatives of  $\Omega_k$  are continuous functions on  $\mathbf{K} \times D^2$ . Then existence of  $b$  follows from compactness of  $\mathbf{K} \times D^2$ .

Take arbitrary  $a \in (0, b)$  and let  $\mathbf{A}_t$  be a homotopy constructed in Lemma 8.2 for  $\Lambda$  and  $a, b$ . We claim that  $\mathbf{A}$  satisfies (8.1) and (8.2).

By (a) of Lemma 8.2  $\mathbf{A}_0 = \omega$ .

Let us prove that  $\mathbf{A}_1(\mathbf{K}) \subset \mathcal{D}^\partial(F)$ , i.e. for each  $k \in \mathbf{K}$  the map  $\mathbf{A}_{1,k} = \Omega'_k$  is a diffeomorphism of  $D^2$  fixed on  $\partial D^2$ . By (b) of Lemma 8.2  $\Omega'_k$  is fixed even on  $D^2 \setminus U_b$ . Moreover by assumption on  $b$  we have that  $\Omega_k : U_b \rightarrow U_b$  is a diffeomorphism, whence by (c) of Lemma 8.2  $A_{t,k} = \mathbf{A}_t(k)$  is also a self-diffeomorphism of  $U_b$  and therefore of all  $D^2$ .

By assumption  $\Omega_k : D^2 \rightarrow D^2$  is a diffeomorphism for all  $k \in \mathbf{L}$ . Then again by (c) of Lemma 8.2  $A_{t,k} = \mathbf{A}_t(k)$  is also a self-diffeomorphism  $D^2$  for all  $k \in \mathbf{L}$ . In other words  $\mathbf{A}_t(\mathbf{L}) \subset \mathcal{D}^+(F)$ .

Finally, the inclusion  $\mathbf{A}_t(\mathbf{P}) \subset \mathcal{E}^\partial(F)$  follows from (8.9) and (e) of Lemma 8.2.

## 9. PROOF OF THEOREM 2.1

First we prove (1) and (2) for the inclusions  $\mathcal{D}^+(F) \subset \mathcal{E}^+(F)$  and  $\mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F)$ . Then we establish (3) and deduce from it (1) and (2) for the inclusion  $\mathcal{D}(F) \subset \mathcal{E}(F)$ .

(1) We have to show that  $\pi_n(\mathcal{E}^r, \mathcal{D}^r) = 0$  for all  $n \geq 0$  if  $r \geq 1$ . Then the result will follow from exact homotopy sequence of the pair  $(\mathcal{E}^r, \mathcal{D}^r)$ .

Let  $\omega : (I^n, \partial I^n) \rightarrow (\mathcal{E}^r, \mathcal{D}^r)$  be a continuous map representing some element of the relative homotopy set  $\pi_n(\mathcal{E}^r, \mathcal{D}^r)$ . Our aim is to show that  $\omega$  is homotopic as a map of pairs to a map into  $\mathcal{D}$ , i.e.  $\omega = 0$  in  $\pi_n(\mathcal{E}^r, \mathcal{D}^r)$ , whence we will get  $\pi_n(\mathcal{E}^r, \mathcal{D}^r) = 0$ .

**Inclusion**  $\mathcal{D}^+(F) \subset \mathcal{E}^+(F)$ . If  $r \geq 1$ , then applying Proposition 8.1 to the case  $\mathbf{K} = I^n$ ,  $\mathbf{L} = \partial I^n$  we obtain that  $\omega$  is homotopic as a map of pairs into  $\mathcal{D}^+(F)$ .

**Inclusion**  $\mathcal{D}^\partial(F) \subset \mathcal{E}^\partial(F)$ . Since

$$(\mathcal{E}^\partial(F), \mathcal{D}^\partial(F)) \subset (\mathcal{E}^+(F), \mathcal{D}^+(F)),$$

we see that  $\omega$  is also an element of  $\pi_n(\mathcal{E}^+(F)^r, \mathcal{D}^+(F)^r)$ , which as just shown is trivial. Then Proposition 8.1 can be applied to the case  $\mathbf{K} = \mathbf{P} = I^n$  and  $\mathbf{L} = \partial I^n$ , and we obtain that  $\omega$  is homotopic as a map of pairs  $(K, L) \rightarrow (\mathcal{E}^\partial(F)^r, \mathcal{D}^\partial(F)^r)$  to a map into  $\mathcal{D}^\partial(F)^r$ . Hence  $\omega = 0$  in  $\pi_n(\mathcal{E}^\partial(F)^r, \mathcal{D}^\partial(F)^r)$ .

(2) We have to show that the map  $\pi_0 \mathcal{D}^0 \rightarrow \pi_0 \mathcal{E}^0$  is surjective for all  $r \geq 0$ . Let  $h \in \mathcal{E}^+(F)$ . It can be regarded as a map from the set  $\mathbf{K}$  consisting of a unique point into  $\mathcal{E}(F)$ :

$$\omega : \mathbf{K} \rightarrow \mathcal{E}, \quad \omega(\mathbf{K}) = h.$$

Then applying Proposition 8.1 we obtain that  $\omega$  is  $C^\infty$ -homotopic to a map into  $\mathcal{D}^\partial(F)$ , whence the inclusion  $\mathcal{D}^\partial(F) \subset \mathcal{E}^+(F)$  yields a surjective map  $\pi_0 \mathcal{D}^\partial(F)^r \rightarrow \pi_0 \mathcal{E}^+(F)^r$  for all  $r \geq 0$ . Therefore in the following diagram induced by inclusions all arrows are surjective:

$$(9.1) \quad \begin{array}{ccc} \pi_0 \mathcal{D}^\partial(F)^r & \longrightarrow & \pi_0 \mathcal{E}^\partial(F)^r \\ \downarrow & & \downarrow \\ \pi_0 \mathcal{D}^+(F)^r & \longrightarrow & \pi_0 \mathcal{E}^+(F)^r \end{array}$$

The proof of the surjectivity  $\pi_0 \mathcal{D}(F)^r \rightarrow \pi_0 \mathcal{E}(F)^r$  is the same as in (1).

(3) It is well known and is easy to prove that *for a topological semigroup  $\mathcal{E}$  the set  $\pi_0 \mathcal{E}$  of path components of  $\mathcal{E}$  admits a natural semigroup structure such that the natural projections  $\mathcal{E} \rightarrow \pi_0 \mathcal{E}$  is a semigroup homomorphism. If  $\mathcal{E}$  is a group, then so is  $\pi_0 \mathcal{E}$ .*

*If  $\mathcal{D} \subset \mathcal{E}$  is a subsemigroup, then the induced map  $\pi_0 \mathcal{D} \rightarrow \pi_0 \mathcal{E}$  is a semigroup homomorphism.*

In our case  $\mathcal{E}^r$  is a topological semigroup and  $\mathcal{D}^r$  is a topological group. From (1) we get that for  $r \geq 1$  the homomorphism  $\pi_0 \mathcal{D} \rightarrow \pi_0 \mathcal{E}$



is a bijection, whence it is a semigroup isomorphism. But  $\pi_0\mathcal{D}$  is a group, whence so is  $\pi_0\mathcal{E}$ .

Let us prove that *all path components of  $\mathcal{E}$  are homeomorphic each other*. By (1) and (2)  $i_0 : \pi_0\mathcal{D} \rightarrow \pi_0\mathcal{E}$  is surjective for any  $W^r$  topologies,  $r \geq 0$ . In particular, this implies that each path component of  $\mathcal{E}$  contains an invertible element. Now the result is implied by the following statement:

**Claim 9.1.** *Let  $\mathcal{E}$  be a topological semigroup such that each path component of  $\mathcal{E}$  contains an invertible element. Then all path components of  $\mathcal{E}$  are homeomorphic each other.*

*Moreover, let  $\mathcal{D}$  be the subgroup consisting of all invertible elements. Then for any two path components  $\mathcal{E}_1$  and  $\mathcal{E}_2$  there exists a homeomorphism  $Q : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that  $Q(\mathcal{E}_1 \cap \mathcal{D}) = \mathcal{E}_2 \cap \mathcal{D}$ .*

*Proof.* Let  $h_1 \in \mathcal{E}_1$  and  $h_2 \in \mathcal{E}_2$  be any invertible elements. Then we can define a homeomorphism  $Q : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  by  $Q(h) = h_2 \cdot h_1^{-1} \cdot h$ . Evidently it is continuous, its inverse is given by  $Q^{-1}(g) = h_1 \cdot h_2^{-1} \cdot g$ , and  $Q(\mathcal{E}_1 \cap \mathcal{D}) = \mathcal{E}_2 \cap \mathcal{D}$ .  $\square$

(1) and (2) for the inclusion  $\mathcal{D}(F) \subset \mathcal{E}(F)$ . Put

$$\mathcal{D}' = \mathcal{D}(F) \setminus \mathcal{D}^+(F), \quad \mathcal{E}' = \mathcal{E}(F) \setminus \mathcal{E}^+(F).$$

Then  $\mathcal{E}'$  consists of full path components of  $\mathcal{E}(F)$  with respect each of  $W^r$  topologies. Hence we have to prove our statement for the inclusion  $\mathcal{D}' \subset \mathcal{E}'$ . We can also assume that  $\mathcal{E}' \neq \emptyset$ .

By Lemma 3.1 there exists  $g \in \mathcal{D}'$ . Hence we can define the following map  $Q : \mathcal{E}' \rightarrow \mathcal{E}^+(F)$  by  $Q(h) = g^{-1} \circ h$  for  $h \in \mathcal{E}'$ . Evidently,  $Q$  is a homeomorphism onto with respect to any of  $W^r$  topologies. Moreover,  $Q(\mathcal{D}') = \mathcal{D}^+(F)$ . Hence  $\pi_n(\mathcal{E}', \mathcal{D}') = \pi_n(\mathcal{E}^+(F), \mathcal{D}^+(F))$ . It remains to note that by  $\pi_n(\mathcal{E}^+(F), \mathcal{D}^+(F)) = 0$  if either  $r \geq 1$  and  $n \geq 0$ , or  $r = 0$  and  $n = 0$ .

## 10. PROOF OF THEOREM 2.3

The proof is similar to the one given in §9. Let

$$\omega : (I^n, \partial I^n) \rightarrow (\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$$

be a continuous map being a representative of some element in the relative homotopy set  $\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$ .

We have to show that  $\omega$  is homotopic as a map of pairs to a map into  $\mathcal{E}^\partial(F)$ . Again we will apply Proposition 8.1 but now the situation is more complicated.

For  $n \neq 1$  denote  $\mathbf{K} = I^n$  and  $\mathbf{P} = \partial I^n$ . Then  $\mathbf{P}$  is path connected and by Proposition 8.1  $\omega$  is homotopic as map of pairs

$$(\mathbf{K}, \mathbf{P}) \rightarrow (\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$$

to a map into  $\mathcal{E}^\partial(F)$ . In this case we can take  $b$  arbitrary, and therefore the arguments hold for the case  $r = 0$  as well. This implies  $\pi_n(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) = 0$  for all  $n \neq 1$  and  $r \geq 0$ .

Suppose  $n = 1$ . Then  $I^1 = [0, 1]$  and  $\partial I^1 = \{0, 1\}$  is not connected, so Proposition 8.1 can be applied only to each of path components  $\{0\}$  and  $\{1\}$  of  $\partial I^1$ , and this is the reason why

$$(10.1) \quad \pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r, \text{id}_{D^2}) \approx \mathbb{Z}.$$

To prove (10.1) use  $0 \in I^1$  and  $\text{id}_{D^2} \in \mathcal{E}^+(F)$  as base points, and thus assume that  $\omega(0) = \text{id}_{D^2}$ . Consider the  $(I^1, r)$ -deformation in  $\mathcal{E}^+(F)^r$  corresponding to  $\omega$ :

$$\Omega : I^1 \times D^2 \rightarrow D^2, \quad \Omega(k, z) = \omega(k)(z).$$

Then  $\Omega_0 = \text{id}_{D^2}$  and therefore the zero function  $\Lambda_0 = 0$  is a shift function for  $\Omega$ . By Lemma 7.2  $\Lambda_0$  extends to a *unique*  $(I^1, r)$ -deformation

$$\Lambda : I^1 \times (D^2 \setminus O) \rightarrow \mathbb{R}$$

being a shift function for  $\Omega$  on  $D^2 \setminus O$  in the sense of that lemma. In particular the last function  $\Lambda_1$  is a shift function for  $\Omega_1 \in \mathcal{E}^\partial(F)$  which is fixed on  $\partial D^2$ . Then by Proposition 8.1 applied to  $\mathbf{P} = \{1\}$ , we get that  $\Lambda_1$  takes constant value on  $\partial D^2$  being an integer multiple of the period of orbit  $\partial D^2$  with respect to  $F$ . Thus

$$\Lambda(\partial D^2) = \rho_\omega \cdot \theta(\partial D^2)$$

for some  $\rho_\omega \in \mathbb{Z}$ . Evidently,  $\rho_\omega$  counts the number of “full rotations” of  $\partial D^2$  during the homotopy  $\Omega$ . We claim that the correspondence  $\rho : \omega \mapsto \rho_\omega$  yields an isomorphism (10.1).

It is easy to see that  $\rho$  induces a *surjective homomorphism*

$$R : \pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r) \rightarrow \mathbb{Z}.$$

To show that  $R$  is a monomorphism suppose that  $\rho(\omega) = 0$ , so  $\Lambda_0(\partial D^2) = \Lambda_1(\partial D^2) = 0$ . Then it follows from (d) of Lemma 8.2 that

$$A(t, 0, z) = A(t, 1, z) = z$$

for all  $t \in I$  and  $z \in \partial D^2$ . In other words,  $\mathbf{A}_t(0), \mathbf{A}_t(1) \in \mathcal{E}^\partial(F)^r$  for all  $t \in I$ . Thus  $\omega$  is homotopic to a map into  $\mathcal{E}^\partial(F)$  via a homotopy relatively  $\partial I^1$ , and therefore it represents a trivial element of  $\pi_1(\mathcal{E}^+(F)^r, \mathcal{E}^\partial(F)^r)$ . This implies that  $R$  is an isomorphism.

## 11. PROOF THEOREM 2.8

Let  $F$  be a TC vector field on  $D^2$  such that  $\nabla F$  is degenerate and  $\ker j \subset Sh(F)$ .

(1) If  $\nabla F = 0$ , then the relation  $Sh(F) = \ker j$  follows from the assumption  $Sh(F) \supset \ker j$  and (1) of Proposition 2.6.

Suppose that  $\{\text{id}\}$  is the path component of  $\text{id}$  in  $L(F)$ . Since  $j$  is continuous from  $W^r$  topology of  $\mathcal{E}(F)$  for  $r \geq 1$ , it follows that  $\mathcal{E}_{\text{id}}(F)^r \subset \ker j = Sh(F) \subset \mathcal{E}_{\text{id}}(F)^r$ . This also implies (2.11).

(2) Suppose that  $\nabla F = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$  for some  $a \neq 0$ . We have to show that  $Sh(F) = \mathcal{E}_{\text{id}}(F)^1 = j^{-1}(\mathcal{A}_{++})$ .

It follows from the definition, see (2.6), that  $\mathcal{A}$  is a group,  $\mathcal{A}_{++}$  is its unity component in  $GL^+(2, \mathbb{R})$ , and  $\mathcal{A}_{--}$ ,  $\mathcal{A}_{+-}$ ,  $\mathcal{A}_{-+}$  are another path components of  $\mathcal{A}$ . Since  $j$  is continuous in  $W^r$  topology of  $\mathcal{E}^+(F)$  for  $r \geq 1$ , it follows that inverse images of these path components are open-closed in  $\mathcal{E}(F)$ . On the other hand  $Sh(F)$  is path connected in all  $W^r$ -topologies, as a continuous image of path connected space  $C^\infty(M, \mathbb{R})$ , whence

$$Sh(F) \subset \mathcal{E}_{\text{id}}(F)^1 \subset j^{-1}(\mathcal{A}_{++}).$$

Conversely, let  $h \in j^{-1}(\mathcal{A}_{++})$ , so  $j(h) = \begin{pmatrix} 1 & a\tau \\ 0 & 1 \end{pmatrix}$  for some  $\tau \in \mathbb{R}$ . We have to show that  $h \in Sh(F)$ .

Evidently  $j(h)$  coincides with

$$\exp(\tau \cdot \nabla F) = \exp \begin{pmatrix} 0 & a\tau \\ 0 & 0 \end{pmatrix}.$$

Consider the flow  $(\mathbf{F}_t)$  of  $F$ . Then,  $j(\mathbf{F}_t) = \exp \begin{pmatrix} 0 & at \\ 0 & 0 \end{pmatrix}$  for all  $t \in \mathbb{R}$ . Hence  $j(\mathbf{F}_\tau) = j(h)$ .

Define the map  $g : D^2 \rightarrow D^2$  by  $g(z) = \mathbf{F}(h(z), -\tau) = \mathbf{F}_{-\tau} \circ h$ . Then  $j(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , i.e.  $g \in \ker j \subset Sh(F)$ .

In other words,  $g(z) = \mathbf{F}(z, \alpha(z))$  for some  $\alpha \in C^\infty(D^2, \mathbb{R})$ . Put  $\beta(z) = \alpha(z) + \tau$ . Then  $h(z) = \mathbf{F}(z, \beta(z))$ , i.e.  $h \in Sh(F)$ .

(3) Due to (2.2) we have only to show that the mapping

$$i_1 : \pi_1 \mathcal{E}^\partial(F)^r \rightarrow \pi_1 \mathcal{E}^+(F)^r$$

induced by the inclusion is an isomorphism. Moreover, by exactness of the sequence (2.3) it remains to show that  $i_1$  is surjective.

Let  $\omega : I \rightarrow \mathcal{E}^+(F)^r$  be a continuous map representing a loop in  $\mathcal{E}^+(F)^r$ , i.e.

$$(11.1) \quad \omega(0) = \omega(1) = \text{id}_{D^2}.$$

We have to show that  $\omega$  is  $r$ -homotopic relatively  $\partial I$  to a map into  $\mathcal{E}^\partial(F)^r$ .

It follows from (11.1) that  $\omega(I)$  included into  $\mathcal{E}_{\text{id}}^+(F)^r$  which by (1) and (2) coincides with  $Sh(F)$ . Thus  $\omega(I) \subset Sh(F)$ . Moreover,  $\omega(\partial I) \subset \ker j$ . Then by Lemma 7.4  $\omega$  is homotopic to a map into  $\ker j$  relatively  $\partial I$ . Hence we can assume that  $\omega$  is a loop in  $\ker j$ .

Consider the  $(I, r)$ -deformation corresponding to  $\omega$ :

$$\Omega : I \times D^2 \rightarrow D^2, \quad \Omega(t, z) = \omega(t)(z).$$

Then  $\Omega_0 = \Omega_1 = \text{id}_{D^2}$  and  $\Omega_k \in \ker j \subset Sh(F)$  for all  $k \in I$ .

In particular, every  $\Omega_k$  has a  $C^\infty$  shift function  $\Lambda_k : D^2 \rightarrow \mathbb{R}$  defined on all of  $D^2$ . Since  $\nabla F$  is degenerate, we have by Lemma 5.2 that such  $\Lambda_k$  is unique. In particular,  $\Lambda_0 = \Lambda_1 = 0$ .

Then it follows from Lemma 7.3 that the map  $\Lambda : I \times (D^2 \setminus O) \rightarrow \mathbb{R}$  defined by  $\Lambda(k, z) = \Lambda_k(z)$  is a  $(I, r)$ -deformation being a shift function for  $\Omega$ .

Take any  $a, b \in (0, 1)$  such that  $a < b$  and consider the homotopy  $\mathbf{A}_t$  of  $\omega$  into  $\mathcal{E}^\partial(F)$  defined by (8.5). Since  $\Omega(0, z) = \Omega(1, z) = z$  for all  $z \in D^2$ , we obtain from (d) of Proposition 8.1  $\mathbf{A}_t(0, z) = \mathbf{A}_t(1, z) = z$  for all  $t \in I$ . In other words  $\mathbf{A}_t$  is a homotopy relatively  $\partial I$ .

(4) This statement follows from (1)-(3) and the well known fact that any finite subgroup of  $\mathrm{GL}^+(2, \mathbb{R})$  is cyclic.

Theorem 2.8 is completed.

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